Incompressible, Viscous, Swirling Flow through a Nozzle

A. Mager*
The Aerospace Corporation, El Segundo, Calif.

The viscous core of an incompressible, swirling flow through the nozzle is treated by momentum-integral equations. These equations, together with the statement of the conservation of mass in the whole nozzle and the conservation of momentum along the nozzle axis, form a system of four ordinary, nonlinear differential equations. This system of equations is solved for the circumferential and axial velocities both inside and outside the core. This system has a singularity which occurs when the square of the ratio of the circumferential to axial velocities, at core boundary, reaches a certain value depending upon the axial velocity profile. This singularity divides the possible solutions into those in which the core is massflow dominated and those in which the core is swirl dominated. The massflow-dominated core occurs at low swirl and is characterized by high discharge coefficient, which is essentially unaffected by the swirl and by a certain maximum total pressure loss which is inversely dependent on nozzle contraction. The flows with the swirl-dominated core have their discharge strongly throttled by the swirl, have the capability to sustain very large total pressure loss, and usually have regions of reversed axial velocity. The increase of Reynolds number tends to decrease the discharge coefficient, whereas the change from laminar to turbulent flow tends to increase the discharge coefficient. Both of these effects are particularly pronounced in swirl-dominated flows.

Nomenclature

a = normalized core area $(=\delta^2)$
A = normalized nozzle area $(=R^2)$
C_{jk},K_j = variable coefficients ($j = 1,\ldots,4$; $k = a,\ldots,W$)
CD = characteristic difference $[=(S/M)^2 A\, f_{15} - f_{16}]$
D = discharge coefficient (maximum value of M)
Det = determinant
$f_1,\ldots,f_{16}$ = functions of α, β, and (a/A) only
$I_1,\ldots,I_{22}$ = massflow and momentum deficiency integrals
M = massflow coefficient $(=$ massflow$/\pi\bar{R}_t^2\bar{\rho}\bar{Q})$
p = static pressure
P = total pressure
r,z = radial and axial coordinates
R = nozzle wall radius
Re = Reynolds number $(= \bar{Q}\bar{R}_t/\bar{\nu})$
S = swirl coefficient $(= \bar{\Gamma}/\bar{R}_t\bar{Q})$
$\bar{Q}$ = maximum velocity $(= (2\bar{P}_\delta/\bar{\rho})^{1/2})$
u,v,w = flow velocities in cylindrical coordinate system
W = value of w outside the core
α = ratio of axial velocities $(= w_0/W)$
β = $[= a(w_{rr})_0/2W]$
$\bar{\Gamma}$ = circulation constant
δ = core radius
Δ = difference between conditions at core boundary and nozzle axis
η = normalized radial distance $(= r/\delta)$
θ = total axial flux of the momentum deficiency in the axial direction
κ = viscosity ratio (eddy to actual)
ν = kinematic viscosity
ρ = density

***Subscripts*†**

i = initial
0 = on nozzle axis
= at the throat
δ = at core boundary

Superscripts

$(\bar{\ })$ ± dimensional values
$(\ )'$ = differentiation with respect to z

Presented as Paper 70-51 at the AIAA 8th Aerospace Sciences Meeting, New York, January 19–21, 1970; submitted February 19, 1970; revision received November 16, 1970. This work was carried out at the Engineering Science Operations of The Aerospace Corporation under Contract F04701-69-C-0066. The author would like to thank Lander, Holt, and Adams for programing the differential equations subroutines of the Aerospace Remote Computation consoles. Without this assistance this work could not be carried out.

* Vice President and General Manager Engineering Science Operations, The Aerospace Corporation, El Segundo, Calif. Associate Fellow AIAA.

† Coordinate subscripts indicate partial differentiation.

Introduction

THE introduction of swirl ahead of the nozzle throat is currently used or proposed in a variety of practical applications. Most of these applications utilize swirl to regulate the massflow. This is the way swirl is being used in fluidic vortex valves, and this is why it is being considered for application to thrust-modulated rockets. Swirl has also been proposed for other, diverse purposes, such as reversing the thrust of jet airplanes,[1] containing radioactive fuel inside advanced nuclear rockets, and stabilizing arcs in electrothermal thrusters.

The inviscid incompressible swirling flow through a nozzle was first solved by Binnie and Hookings[2] who assumed that the flow is potential and thus obtained a core void of any fluid. Later, in a series of experiments with water in transparent nozzles, Binnie et al.[3] have shown that, when the cavity exists, this potential solution does describe with reasonable accuracy the variation of massflow with swirl. However, they also noted that, at low swirl, core void of water did not form on its own accord and had to be coaxed by artificial air injection. This injection also had a strong stabilizing effect on the flow which, without the injection, was violently oscillatory. In spite of this artificial establishment of the core, the agreement between the theoretically predicted and the measured mass flows continued to be good. At still lower swirl, though the flow was becalmed, void core could not be coaxed by injection, and the check of the theory with experiment regarding the throttling effect of the swirl could not be made.

To avoid the formation of the empty core, Fraenkel[4] proposed a solution of this problem which assumed that the swirl caused the fluid to rotate like a solid body. He found that this solution became unacceptable because it altered the assumed upstream boundary conditions whenever the ratio of the initial tangential velocity at nozzle wall to the initial axial velocity exceeded the value of 3.83/2. Although this alteration of the upstream boundary conditions could be avoided (by adding a cancelling upstream wave) the so-modified solution produced reverse flow and standing waves in the nozzle. It has been long realized from various experiments on vortices confined in tubes that the solid body rotation caused by the swirl occurs only in a core of the flow, close to the axis, and that the remainder of the fluid tends to rotate like a potential vortex. King,[5] who also worked on the inviscid, swirling flow through the nozzle, was the first to construct an analytical model employing such a mixture of solid body and potential rotations. His solutions also produced reverse flow and standing waves in the core whenever Fraenkel's ratio (but now evaluated at the edge of the core) exceeded 3.83/2, and the initial core diameter was large enough. King also suggested that these unusual nozzle flows may be connected with the phenomenon of sudden vortex expansion (also known as vortex breakdown) because this expansion, as proposed by Gartshore,[6] occurs, not as a result of unbounded growth of the initially small disturbances, but because of the diffusion of vorticity accompanying the axial flow reversal. In fact, Gartshore, who studied the viscous vortex embedded in uniform axial flow, has shown that flow reversal in his solutions was dependent on the ratio of the axial to angular momentum deficiency fluxes which, in turn, could be related to the Fraenkel's value of 3.83/2.

The work of Benjamin[7] has an important bearing on the ideas of King and Gartshore. Benjamin showed that in swirling flow there may be two conjugate states of flow: one, at low swirl, unable to support standing waves of finite length, which he calls "supercritical" and another, with high swirl, in which the standing waves can occur, which he names "subcritical." The sudden vortex expansion is then postulated to be the transition from one state to another. Since the "flow force" (e.g., the sum of the axial momentum flux and pressure) for the subcritical state is greater than for the supercritical state (at the same total pressure), transitions from one state to another are possible only when this difference can be made up either by wave resistance or, for large differences, by a dissipative process which substantially changes the total pressure. Thus, after transition, the subcritical state may have large total pressure loss. In addition, Benjamin points out, the subcritical state solutions allow for regions of reversed flow (in which viscosity may play an important role) and also regions of cavitating flow.

Benjamin's postulate that the transition from one conjugate state to the other represents the phenomenon of vortex breakdown has been recently challenged by Bossel.[8] Bossel demonstrated that flows similar to those obtained in vortex breakdown can be computed by considering only one of the conjugates. However, regardless of whether the vortex breakdown is the transition between the two states of flow, the fact that two separate states exist in swirling flow through the nozzle, and that the changeover between them is sudden, has been demonstrated experimentally by Iserland.[1] He found, when increasing either the swirl magnitude or the nozzle diameter, that the flow was little affected at first. However, upon reaching a certain critical value of the swirl or nozzle diameter, an additional minimal increase of these quantities triggered a sudden drastic change in the nature of the flow. By a series of measurements carried out at nozzle exit, Iserland was then able to demonstrate that the "subcritical" flow‡ was one with a "full" core, whereas the "supercritical" flow had a core which was essentially void of energy (total pressure equal to exit ambient pressure), and thus he called this core a "hole." Moreover, the supercritical flow had the general characteristics of the empty-core solutions; that is, it was strongly throttled by the swirl and had a much larger exit divergence angle than the subcritical flow. Unfortunately, since Iserland gave but indirect measure of the swirl magnitude, it is impossible to quantitatively compare his results with theory.

It is apparent from the preceding discussion that in the core of the swirling flow, particularly when the velocity gradients are large, the viscosity may play an important dissipative role. In an effort to clarify this role, the present paper formulates the problem of the incompressible, swirling flow through the nozzle by assuming that the flow outside the core is potential and inside the core is viscous. The solution to both of these flows is obtained simultaneously, for an initially prescribed nozzle shape, by numerically integrating a set of four, nonlinear total differential equations. To avoid any controversy regarding the true nature of the turbulent shear, the treatment is purposely restricted to laminar flow, although the equations in their integrated form would easily permit the use of the apparent (eddy) viscosity. Hopefully, despite swirling flows often being turbulent, the study of laminar flow will afford us enough insight to increase our understanding of the process that takes place. Still, to give the reader some idea of the gross changes that may occur, a crude assessment of the effect of the turbulent flow on the discharge coefficient is also made.

‡ Though Iserland uses the terms "critical," "subcritical," and "supercritical," it appears that the meaning of his terms is exactly reversed from that of Benjamin. In spite of this we follow his usage here.

Momentum Integral Equations

We start our development by dividing all velocities by the maximum velocity $\bar{Q}$, all pressures by the maximum dynamic pressure $0.5\,\bar{\rho}\bar{Q}^2$, and all lengths by the nozzle throat radius $\bar{R}_t$. If the so-nondimensionalized quantities are used, the conservation of mass and momentum for axially symmetric, incompressible flow, with gradients in the axial direction much smaller than those in the radial direction, may be written as§

$$(ur)_r + (wr)_z = 0 \tag{1a}$$

$$v^2/r = \tfrac{1}{2}p_r \tag{1b}$$

$$(uvr^2)_r + (vwr^2)_z = (1/Re)[r^2(v_r - v/r)]_r \tag{1c}$$

$$(uwr)_r + (w^2r)_z = -(r/2)p_z + (1/Re)[r(w)_r]_r \tag{1d}$$

where the Reynolds numbers is $Re = (\bar{Q}\bar{R}_t/\bar{\nu})$.

Since the velocities at the boundary of the core are assumed to change in a continuous manner into the outer potential flow, and since the shear forces vanish on the axis, the boundary conditions are

$$\text{at } r = 0\text{:}\quad u = v = w_r = 0 \tag{2a}$$

$$\text{at } r = \delta\text{:}\quad w = W(z),\quad v = (S/\delta),\quad p = p_\delta,\quad w_r = (rv)_r = 0 \tag{2b}$$

where the constant swirl coefficient is defined as $S \equiv (\bar{\Gamma}/\bar{R}_t\bar{Q})$.

Next, Eqs. (1a) and (1b) can be used to eliminate the velocity u and the pressure p, respectively, so that the further integration of Eqs. (1c) and (1d) with $a \equiv \delta^2$, yields after considerable manipulation

$$I_{12}' = (2/Re) \tag{3a}$$

$$I_{11}' + \tfrac{1}{2}I_{22}' + I_1W' - (S/2)^2(a'/a) = 0 \tag{3b}$$

where

$$I_1 \equiv \int_0^\delta (W - w)rdr;\quad I_{11} \equiv \int_0^\delta w(W - w)rdr$$

$$I_{12} \equiv \int_0^\delta w\left[1 - \left(\frac{rv}{S}\right)\right]rdr;\quad I_{22} \equiv \int_0^\delta v^2rdr$$

Furthermore, at $r = 0$ Eq. (1d) gives the variation of total

§ All velocities and coordinates are schematically illustrated in Fig. 1.

pressure P_0

$$P_0' = 2(w_0 w_0' - WW') + (S/a)^2 a' - (\Delta p)' = (4/Re)(w_{rr})_0 \quad (3c)$$

with

$$\Delta p \equiv p_\delta - p_0 = 2 \int_0^\delta \frac{v^2}{r} dr$$

and the conservation of massflow within the whole nozzle may be written as

$$M = WA - 2I_1 \quad (3d)$$

with the constant massflow coefficient defined as $M \equiv (\text{massflow}/\pi \bar{R}_t^2 \bar{\rho} \bar{Q})$.

The quantities I_1, I_{11}, I_{12}, and I_{22} are closely related to the usual displacement and momentum thicknesses of the three-dimensional boundary-layer theory. In particular, I_{12} is a measure of the total deficiency of the angular momentum which is transferred by the velocity w along the nozzle axis. Since Eq. (3a) may be integrated to give

$$I_{12} = I_{12,i} + 2z/Re \quad (4a)$$

we see that the change in this deficiency is directly proportional to the distance along the axis and that this change tends to disappear for very large Reynolds numbers.

Equation (3b) is in general not integrable in closed form unless I_1 or A are known functions of W. However, when W is a constant, one gets a solution identical to that of Gartshore[9]

$$I_{11} + 0.5\, I_{22} - 0.25\, S^2 \ln a = \text{const} = \theta \quad (4b)$$

Similarly, when the flow takes place in a constant area duct, one obtains from Eq. (3d) $W' = (2/A_i)I_1'$ which yields

$$I_{11} + (I_{22}/2) + (I_1^2/A_i) - (S/2)^2 \ln a = \theta \quad (4c)$$

The constants $I_{12,i}$ and θ, which represent the initial values of the conserved quantities, have been shown by Gartshore to be important in determining the nature of the solutions. In particular, for a viscous vortex in still air, whenever the ratio $\theta/(I_{12,i}W)^{2/3}$ reached a certain value, Gartshore obtained reversed flow solutions.

Equations (3a) through (3d) determine four functions of z which, we hope, will properly describe the variation of the velocities v and w. To this end we represent the velocities v and w by polynomials in $\eta(=r/\delta)$ with the coefficients of these polynomials chosen to satisfy the boundary conditions at $r = 0$ and $r = \delta$ given by Eq. (2). After some elementary matching one obtains

$$v = (S/\delta)(2\eta - \eta^3) \quad (5a)$$

$$w = W\{\alpha + \beta\eta^2 + [4(1 - \alpha) - 2\beta]\eta^3 + [\beta - 3(1 - \alpha)]\eta^4\} \quad (5b)$$

where $\alpha \equiv (w_0/W)$ and $\beta \equiv [a(w_{rr})_0/2W]$. Using these ex-

Table 1 Coefficients $C_{j,k}$ and K_j

	$C_{j,k}$				
↓j \ k→	a	α	β	W	K_j
1	Wf_2	$29Wa/252$	$1.9Wa/252$	af_2	$2/Re$
2	$W^2f_3-(S^2/4a)$	$W^2af_4/15$	$-W^2af_5/1260$	$Wa(2f_3+f_1)$	0
3	$-2Wf_1$	$2WA/5$	$Wa/30$	$-2af_1+A$	$-WA'$
4	$(5/3)(S/a)^2$	$W^2\alpha$	0	$W(\alpha^2-1)$	$(4\beta W/Rea)$

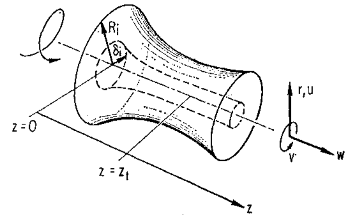

Fig. 1 Velocities and coordinates.

pressions in the definition of the various integrals, one finds

$$I_1 = aWf_1; \quad I_{12} = aWf_2; \quad I_{11} = aW^2f_3$$
$$I_{22} = (11S^2/24); \quad \Delta p = (7S^2/3a) \quad (6)$$

where f_1, f_2, and f_3 are functions of α and β only and are given in the Appendix. A further substitution into Eqs. (3a, 3b, and 3c) and the differentiated form of Eq. (3d) gives a system of four, nonlinear, differential equations which may be written as

$$C_{ja}a' + C_{j\alpha}\alpha' + C_{j\beta}\beta' + C_{jW}W' = K_j \quad (j = 1, \ldots, 4) \quad (7)$$

with the coefficients $C_{jk}(k = a, \ldots, W)$ and K_j listed in Table 1. These equations are to be solved with the known initial conditions at $z = 0$: $a = a_i$, $\alpha = \alpha_i$, $\beta = \beta_i$, $W = W_i$, and an a priori selected set of values for the parameters Re, S, and M.

Once a and W are known, the pressure in the core may easily be found. In particular, since $P_\delta = 1$, the pressure is given by

$$p = 1 - W^2 + (S^2/a)[4\eta^2 - 2\eta^4 + (\eta^6/3) - (\tfrac{10}{3})]$$

Singularity of the System

The system of Eqs. (7) can be solved for the derivatives a', α', β', and W' providing the determinant of the coefficients C_{jk} does not vanish. The conditions which cause such vanishing were evaluated by expanding this determinant which thus was found to equal

$$\text{Det}\, C_{jk} = (aW^5A/22680)[(S^2/aW^2)f_{13} - f_{14}] \quad (8a)$$

with f_{13} and f_{14} functions of a/A, α and β only (see the Appendix). A singularity in the system of Eqs. (7) will therefore occur whenever $(S^2/aW^2) = f_{14}/f_{13}$, providing, of course, the numerators of the derivatives a', α', β', and W' are different from zero. The simultaneous vanishing of these numerators was found by expansion of the appropriate determinants and a numerical solution to occur only when there is a stagnation point at the throat (e.g., when $A' = \alpha = 0$, $\beta = 1$, $a/A = 0.4709085$, $(S^2/aW^2) = 0.152602$). This condition, known as the critical point solution,¶ also satisfies $(S^2/aW^2) = f_{14}/f_{13}$ so that it leads to the indeterminate form 0/0. Clearly, although such an indeterminate form may be of some significance, it constitutes nevertheless a very special case which normally does not occur. Thus, in general, we may expect a singularity whenever $(S^2/aW^2) = f_{14}/f_{13}$, and to avoid it one must have (S^2/aW^2) either sufficiently large or sufficiently small for the Det C_{jk} to be either positive or negative in the entire nozzle. One easily recognizes that (S^2/aW^2) represents the square of the ratio of the circumferential and axial velocities at the core boundary. This parameter is thus closely related to the Fraenkel's and King's numbers, except that now its special value is not a constant but varies with z. Similarly, it is also closely related to the Benjamin's characteristic parameter whose role is to indicate the proximity of transition from one state of flow to its conjugate.

¶ The term critical point is used here in the conventional sense appropriate for a sytem of ordinary differential equations, which, however, is quite different from that of Iserland or Benjamin.

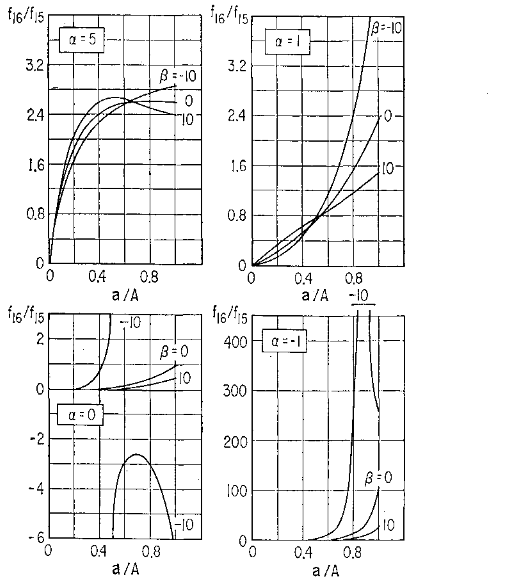

Fig. 2 Singularity condition for various velocity profiles.

As mentioned previously, to avoid the singularity one must have (S^2/aW^2) either sufficiently large, or sufficiently small in the entire nozzle. To enhance the likelihood of making the proper choices at the start of the integration process it is convenient to modify the expression for the determinant of C_{jk} by use of Eq. (3d)

$$CD \equiv 22680\ (\mathrm{Det}\ C_{jk}/A^2W^5) = A(S/M)^2f_{15} - f_{16} \quad \text{(8b)}$$

so that for the occurrence of singularity $A(S/M)^2$ must equal f_{16}/f_{15}. Since the swirl and massflow coefficients are constants, $A(S/M)^2$ varies with nozzle cross-sectional area and the problem is: to choose the swirl and massflow coefficients so that CD remains either positive, or negative everywhere in the nozzle. Now suppose that the velocity profiles (e.g., f_{16}/f_{15}) were reasonably invariant and that the massflow were chosen to be larger than that giving $CD = 0$ at the inlet. Consequently CD would be initially negative and because of the nozzle contraction it would continue to decrease downstream of the inlet, thus shifting the state of the flow farther away from the singularity. One may think of this type of nozzle flow as *massflow-dominated.* Alternatively, if f_{16}/f_{15} were again invariant but the swirl were chosen to be larger than that which would produce $CD = 0$ at the inlet, though the initial CD would be positive, downstream from the inlet its value would decrease towards zero thereby bringing the state of the flow closer to the singularity. One may think of this type of nozzle flow as *swirl-dominated.* It is clear from this simplified picture that in order to avoid the singularity in such a flow one must choose (S/M) large enough to at least overcome the effect of nozzle contraction.

In the actual solutions f_{16}/f_{15} is not invariant; however, these simplified considerations permit an insight into the manner in which the choices of S and M should be made to avoid the singularity. Nevertheless, to complete this picture one needs to indicate the sensitivity of f_{16}/f_{15} to the various changes of the velocity profile that can take place. To do this, the values of f_{16}/f_{15} were computed for various assumed α, β, and a/A. The results of such computations are shown in Fig. 2. As may be seen from this figure, for positive α the ratio f_{16}/f_{15} behaves quite regularly. For negative α, however, that is when the core flow is in the direction opposite from that in the freestream, the ratio f_{16}/f_{15} tends to acquire large poles and discontinuities. The existence of these irregularities implies that the core with reversed flow is strongly sensitive to slight variations of the velocity profile and that such variations may easily trigger the occurrence of singularity in such flows. Also, since $A(S/M)^2$ cannot be negative, one may see in Fig. 2 that there are some velocity profiles for which the singularity cannot occur.

Numerical Solutions

General

The numerical solutions were carried out on a remote console connected to the CDC 6400 computer using a differential equations subroutine. This subroutine employs matrix inversion and the Kutta-Runge integration to solve a system of ordinary differential equations of the first order. The accuracy of the computations was assured by use of very small steps and was continually checked by determining that the difference between the integrated and initial values of M and $I_{12,i}$ is zero. Nearly all of the computations were carried out for nozzles whose shape is given by $A = A_i + (z/z_t)(A_i - 1)[(z/z_t) - 2]$ with $z_t = 30$, $A_i = 5$ using $Re = 10{,}000$ and a uniform initial velocity profile (e.g., $\alpha_i = 1$ and $\beta_i = 0$). These conditions will be designated as standard. For the designation of the remaining two initial conditions and the swirl S, usually $(a/A)_i$ was first specified so that f_{15} and f_{16} could be found and then ΔP_i** or S, and CD_i or M, were fixed.

Massflow-Dominated Core

In the discussion on singularity it was pointed out that if the velocity profile were invariant, one could insure that the flow in the entire nozzle would be massflow dominated by choosing CD_i negative. The computations which we have performed indicate that in general, for massflow-dominated flow, the variation of the velocity profile is not strong enough to overcome this effect of nozzle area and therefore the flow tends to remain massflow dominated whenever CD_i is but slightly smaller than zero. This fact then specifies that $(S/M)^2$ must be smaller than $f_{16,i}/(Af_{15})_i$, which for uniform velocity profile means generally a low swirl and a high mass flow.

A typical solution at standard conditions is illustrated in Fig. 3. As may be seen from this figure, the initially uniform axial velocity soon develops a maximum on the axis, and this maximum persists throughout the nozzle, although in the passage through the throat there is a definite tendency for the velocity profile to become uniform again. This maximum notwithstanding, the difference in total pressure ΔP remains essentially constant throughout the nozzle. On the other hand, the core area, the axial velocity outside the core and the

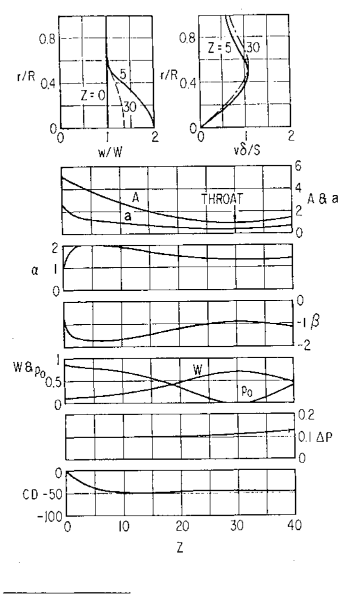

Fig. 3 Solution with a massflow - dominated core at standard conditions and $S = 0.2773$, $M = 0.73$, $\Delta P_i = 0.1025$.

** $\Delta P \equiv 1 - P_o = W^2(1 - \alpha^2) + S^2/(0.3a)$.

static pressure p_0 all have an extremum near the throat. Since the pressure p_0 has a minimum there, one may, by suitable adjustment of the initial conditions, obtain $p_{0t} = 0$. The value of M corresponding to this special case will henceforth be designated D — the discharge coefficient. Any increase of massflow beyond D shifts the point at which $p_0 = 0$ upstream of the throat and leads to $p_{0t} < 0$. Since this is an inadmissible situation,†† D represents the maximum attainable massflow for a given swirl and a given set of initial conditions α_i, β_i, and a_i. It is thus clear that even for initially uniform flow (e.g., $\alpha_i = 1$, $\beta_i = 0$) and fixed Re, D remains a function of both S and a_i (or alternately S and ΔP_i) and cannot be represented, as in the potential flow solution, as a function of S alone.

Another quantity which differs significantly from the potential solution for swirling flow is the velocity W which, as was previously mentioned, has a maximum near the throat. This maximum is similar to the one which would be obtained for nonswirling one-dimensional flow. In contrast, the potential solution for swirling flow requires that $(W')_t \neq 0$ and therefore W, as given by such a solution, has to keep increasing continuously through the nozzle.

Since the maximum possible value of $(S^2/aW^2)_i$ must be smaller than f_{14}/f_{13}, the maximum possible value of the initial total pressure difference is limited by $(\Delta P_i)_{max} = \{W^2[1 - \alpha^2 + (f_{14}/0.3f_{13})]\}_i$ which for uniform initial velocity profile becomes $(\Delta P_i)_{max} = (D^2f_{14}/0.3A^2f_{13})_i$ so that for each $(a/A)_i$ (or for each S) there is a maximum ΔP_i. This maximum ΔP_i forms one boundary of Fig. 4 where the complete domain of massflow-dominated cores, evaluated at standard conditions, is shown. The other boundary is established by the line $(a/A)_i = 1$ because the present equations do not allow cores larger than the nozzle itself. Of course, in reality, the viscous effects on nozzle walls will even further limit the solutions to values of $(a/A)_i$ which are somewhat lower than unity.

As can be seen from this figure, for standard conditions the massflow-dominated core cannot sustain very large total pressure loss. This maximum value of ΔP_i (or for that matter, ΔP—since, as previously mentioned, the total pressure loss tends to remain invariant throughout the nozzle) will be even smaller for nozzles with large contraction ratio A_i and for certain nonuniform velocity profiles.

Figure 4 also shows the lines of constant D plotted as functions of S and ΔP_i. Since the lowest value of D is $D = 0.7$, it is obvious that the swirl does not have a strong throttling effect on a flow with a massflow-dominated core. This finding is in a qualitative agreement with the experimental results of Iserland who also observed but a small effect on the massflow at low swirl.

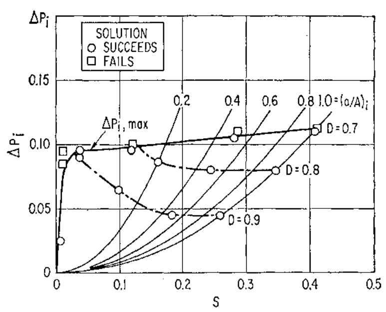

Fig. 4 Domain of solutions with massflow-dominated core at standard conditions.

†† All pressures are absolute. Experimentally, $p_{0t} = 0$ will not occur because of such effects as cavitation. In practice, therefore, it will be convenient to define some absolute, minimum pressure (say vapor pressure) as the reference from which all other pressures will be measured. With such a reference $p_{0t} = 0$ will still signify that all of the attainable conversion of total pressure P_{0t} into dynamic pressure w_{0t}^2 has taken place and therefore values of $p_{0t} < 0$ are not possible.

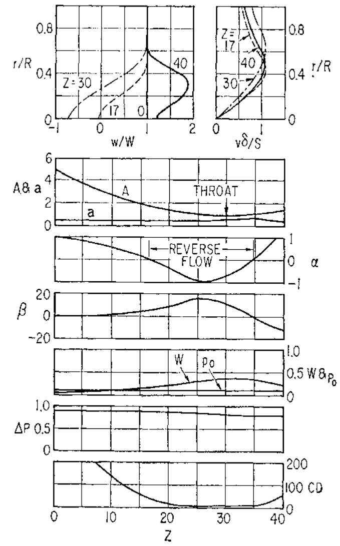

Fig. 5 Solution with a swirl-dominated core at standard conditions and $S = 0.3674$, $M = 0.30$, $\Delta P_i = 0.90$.

Swirl-Dominated Core

The solutions of a swirl-dominated core are somewhat more difficult to obtain than those for a massflow-dominated core. This is so because CD decreases with nozzle area so that its lowest value tends to occur near the nozzle throat. When starting a solution, one is thus not certain whether the initially chosen $(S/M)^2$ is large enough to insure positive values of CD in spite of the decreasing area. This initial uncertainty is made even more acute because the swirl-dominated core often contains reversed flow and thus, as was already discussed in connection with Fig. 2b, it can be drastically affected by slight changes of the velocity profile.

In view of these uncertainties, the procedure which we adopted was to choose, for initially uniform flow, some value of $(a/A)_i$ and ΔP_i, which permitted the determination of S, $f_{15,i}$, and $f_{16,i}$. Next, CD_i was progressively increased by successively lower choices of the massflow M until an acceptable solution with swirl-dominated core extending through the entire nozzle was obtained. The highest value of M at which such solution existed was again defined as the discharge coefficient D. Notice here that, as in the case of massflow-dominated core, for M slightly higher than D the flow was usually free of singularity but showed regions of $p_0 < 0$ which often extended through nearly the whole nozzle. These regions could be interpreted as a possible indication of flow cavities which are often observed in swirling, incompressible flow. However, since the present equations have improper boundary conditions for such cavities, the solutions which contained regions of $p_0 < 0$ were excluded from present considerations, as though they had the singularity.

An example of a typical solution with a swirl-dominated core is shown in Fig. 5. Perhaps the most striking feature of this example is the reversed flow which is indicated by the negative values of α and which occurs despite the static pressure on the nozzle axis being essentially constant throughout the nozzle. Consistent with the reversed flow, the total pressure loss ΔP shows a definite decrease as the flow progresses towards the nozzle outlet. Outside of the core, whose area remains relatively invariant, the axial velocity W shows a maximum located well past the throat.

It is thus apparent that the solutions for swirl-dominated core show a fundamentally different behavior from those for the massflow-dominated core. This difference becomes even more pronounced when the magnitude and variation of the discharge coefficients for the two families of solutions is compared. This is illustrated in Fig. 6 which shows D as a function of S on lines of constant ΔP_i for both kinds of solution. As may be seen from this figure, the swirl-dominated core, which can sustain very large total pressure loss, has not only lower discharge coefficients but is also strongly throttled

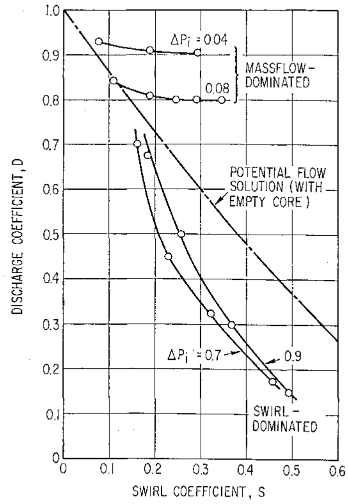

Fig. 6 Discharge coefficients at standard conditions.

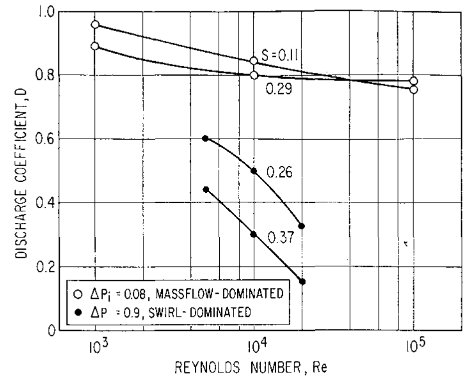

Fig. 7 Effect of Reynolds number.

by the swirl. In fact, the throttling effect of the swirl on the flow with the swirl-dominated core roughly parallels that expected from the solution for swirling, potential flow with an empty core. On the other hand, the massflow-dominated cores have discharge coefficients which are not only larger than those of the swirling potential flow but also are essentially unaffected by the swirl.

These results appear to be in a qualitative agreement with the experimental data of Iserland[1] who, as was discussed in the Introduction, observed the strong throttling action of the swirl to commence suddenly upon reaching a certain critical value of the swirl and/or nozzle area. Notice that the increase of both of these quantities will also increase our $(S/M)^2A$ (and thus CD) so that Iserland's change from "subcritical" flow to "supercritical" flow could well correspond to a sudden changeover from massflow-dominated core to a swirl-dominated core. Although the actual process of such a changeover has not been studied in this report, it is clear that there are enough similarities to suggest this correspondence between Iserland's and our flows. In fact, this correspondence may also be applicable to the change from "supercritical" to "subcritical" flows of Benjamin because his changeover is also dependent on a parameter related to $(S/\delta W)$ and because his "subcritical" flow has similar characteristics to our flow with swirl-dominated core.

Effect of Reynolds Number

The effect of Reynolds number variation on massflow-dominated core is shown in Fig. 7. In general, a decrease of Reynolds number tends to increase the core area and thus reduces the axial velocity w_0 and increases p_0. To compensate for this, one needs to increase the massflow at which $p_{0t} = 0$. The discharge coefficient D is thus increased. Of course, the reverse of this process occurs when the Reynolds number is increased and the discharge coefficient is reduced.

Similar considerations apply also to the swirl-dominated core except that now the flow is much more sensitive to slight changes in the velocity profile so that large changes of Reynolds number may not be achievable in a flow from which singularities are absent. In fact, in our computations we were unable to find a solution for swirl-dominated core with $\Delta P_i = 0.9$ and $Re = 100{,}000$. For $Re = 20{,}000$ and $Re = 5000$, however, the trends shown in Fig. 7, aside from the aforementioned increased sensitivity, are similar to those for the massflow-dominated core.

Figure 7 may also be used for an admittedly imprecise assessment of the effect of the turbulent flow. The effect of turbulence has been approximated by a number of investigators through a simple replacement of the actual viscosity by a constant eddy viscosity in the laminar form of Eqs. (1a–1d). According to Hall[10] the ratio of this eddy viscosity to the actual viscosity has been shown by Owen to remain roughly constant when $\overline{\Gamma}/\bar{\nu}$ is invariant. Consequently, assuming that for a given S-line the value of this ratio is κ and that the actual Reynolds number is Re, then the appropriate discharge coefficient for the turbulent flow can be found in Fig. 7 at the Reynolds number $= Re/\kappa$. Since κ is generally by one to three orders of magnitude larger than unity, the general shape of curves in Fig. 7 indicates that the effect of turbulence is to increase the discharge coefficient and that this increase will be particularly pronounced in swirl-dominated flows.

Appendix

Functions of α and β

$$f_1 = \tfrac{1}{5}[1 - \alpha - (\beta/12)], \quad f_2 = \tfrac{1}{252}(13 + 29\alpha + 1.9\beta)$$

$$f_3 = \tfrac{1}{15}[1 + \alpha - 2\alpha^2 - \tfrac{1}{84}(23\alpha\beta + \beta^2 - 2\beta)]$$

$$f_4 = 1 - 4\alpha - \tfrac{23}{84}\beta, \quad f_5 = -2 + 23\alpha + 2\beta,$$

$$f_6 = f_4 + \tfrac{1}{7}f_5, \quad f_7 = f_1 + f_3, \quad f_8 = f_4 + \tfrac{145}{798}f_5$$

Functions of a/A, α and β

$$f_9 = 252\, f_2\{3\alpha[(f_5/42) + (af_7/A)] - [(\alpha^2 - 1)af_6/(5A)]\}$$

$$f_{10} = (a/A)(\alpha^2 - 1)[(2697f_3/145) - (114\, f_1f_8/5)] + 171\alpha[f_3 + (2af_1f_7/A)]$$

$$f_{11} = (a/A)\{(\alpha^2 - 1)(26977/580) + 31(f_3 + f_7) - 84[\tfrac{9}{4}\alpha + f_6]f_2\}$$

$$f_{12} = 19[\tfrac{9}{4}\alpha + f_8][1 - 2(a/A)f_1], \quad f_{13} = f_{11} + f_{12}$$

$$f_{14} = f_9 + f_{10}, \quad f_{15} = [1 - 2(a/A)f_1]^2f_{13}, \quad f_{16} = (a/A)f_{14}$$

References

[1] Iserland, K., "Untersuchungen über die Umlenkung eines freien Luftstrahls mit Hilfe von Drall," *Mitteilungen aus dem Institut für Aerodynamik an der Eidgenössischen Technischen Hochschule in Zürich* Nr. 25, 1958.

[2] Binnie, A. M. and Hookings, G. A., "Laboratory Experiments on Whirlpools," *Proceedings of the Royal Society, Ser. A*, Vol. 194, 1948, pp. 398–415.

[3] Binnie, A. M., Hookings, G. A., and Kamel, M. Y. M., "The Flow of Swirling Water through a Convergent-Divergent Nozzle," *Journal of Fluid Mechanics*, Vol. 3, Pt. 3, 1957, pp. 261–274.

[4] Fraenkel, L. E., "On the Flow of Rotating Fluid Past Bodies in a Pipe," *Proceedings of the Royal Society, Ser. A*, Vol. 233, 1956, pp. 506–526.

[5] King, W. S., "A Theoretical Investigation of Swirling Flows through a Nozzle," Ph.D. dissertation, 1967, Univ. of California, Los Angeles.

[6] Gartshore, I. S., "Recent Work in Swirling Incompressible

Flow," Aeronautical Report LR-343 N.R.C. 6968, 1962, National Research Council of Canada.

[7] Benjamin, B. T., "Theory of the Vortex Breakdown Phenomenon," *Journal of Fluid Mechanics*, Vol. 14, 1962, pp. 593–629.

[8] Bossel, H. H., "Vortex Breakdown Flowfield," *The Physics of Fluids*, Vol. 12, 1969, pp. 498–508.

[9] Gartshore, I. S., "Some Numerical Solutions for the Viscous Core of an Irrotational Vortex," Aeronautical Report LR-378, N.R.C. 7479, 1963, National Research Council of Canada.

[10] Hall, M. G., "The Structure of Concentrated Vortex Cores," *Progress in Aeronautical Sciences*, edited by D. Küchemann, Vol. 7, Pergamon Press, New York, 1966, pp. 53–110.

APRIL 1971 AIAA JOURNAL VOL. 9, NO. 4

Evolution of the Laminar Wake behind a Flat Plate and Its Upstream Influence

L. I. SCHNEIDER*
North American Rockwell Corporation/Space Division, Downey, Calif.

AND

V. E. DENNY†
University of California, Los Angeles, Calif.

Analysis of laminar, two-dimensional, viscous flow of an incompressible fluid over the trailing edge of a vanishingly thin flat plate is presented. A coordinate transformation is introduced which admits sufficient scaling of the problem to enable detailed study of the separating flow and its interactions with the inviscid stream, as well as with a presumed boundary-layer flow in an intermediate region. Second-order boundary-layer theory is applied in an iterative manner to extract a first approximation to the displacement thickness and associated induced pressure distribution. It is found that the near-wake region is nonisobaric, the classical isobaric result of Blasius-Goldstein for displacement thickness being slightly high just prior to the trailing edge and about 10% low downstream. Local corrections to the velocity and pressure distributions are obtained by numerically solving the full Navier-Stokes equations, using the second-order boundary-layer results to establish Dirichlet boundary conditions over a rectangular region enclosing the trailing edge point. It is found that the physical extent of this region is of order $R^{-3/4}$. Within the region, shear at the plate increases with x and becomes very large at the trailing edge. No significant correction to the displacement thickness was found on resolving the second-order boundary-layer problem, using the Navier-Stokes results as inner boundary conditions.

Nomenclature

c_f = drag coefficient, $(2/R)(\partial u/\partial y)|_{y=0}$
c_p = pressure coefficient, $(P - P_\infty)/\rho U_\infty^2$
L = plate length, ft
m = continuity equation, $u_x + v_y$
p = pressure
P = pressure, $lb_m/ft\ sec^2$
r = polar radius
R = Reynolds number, LU_∞/ν
s,t = Cartesian coordinates
u,v = velocity components
U = velocity component, fps
x,y = Cartesian coordinates
α = relaxational parameter
δ^* = displacement thickness
θ = polar angle
λ = integral averaging parameter for difference analogues to equations
ν = kinematic viscosity, ft^2/sec
ρ = density, lb_m/ft^3

Subscripts

e = at the boundary-layer edge
i,j = ijth node point
∞ = at infinity

Received October 16, 1970; revision received September 8, 1970.

* Project Engineer. Member AIAA.

† Assistant Professor, Department of Energy and Kinetics.

1. Introduction

RECENTLY, Plotkin and Flügge-Lotz,[1] using finite difference methods, obtained an improved first approximation to the solution of the trailing edge problem at large Reynolds numbers. Their principal objective was to bridge the gap between the Blasius solution upstream of the trailing edge and the near-wake solution of Goldstein.[2] They solved the Navier-Stokes equations over a region, roughly centered on the trailing edge, whose thickness is several Blasius boundary layers in extent. Dirichlet boundary conditions for the resulting elliptic problem were extracted from the Blasius solution upstream, from the Goldstein solution downstream, and by presuming that the freestream conditions at infinity apply at the region's edge. Thus, they neglect the effect of displacement thickness on the streamwise pressure distribution. Their results indicate existence of a singularity at the trailing edge.

Earlier, several investigators studied the problem using analytical techniques to investigate the full Navier-Stokes equations. The earliest reference appears to be due to Lykoudis,[3] who concluded that the domain of upstream influence of the trailing edge (say Δ) is of the order of a boundary-layer thickness based on the length δ, which in turn is the boundary-layer thickness based on length L measured from the leading edge. Thus, in analogy with ordinary boundary-